

Julia and Fatou Set

Cor. If $z_0 \in J(f)$, then $\{z \in \hat{\mathbb{C}} \mid f^n(z) = z_0\}$ is dense in $J(f)$.

Proof. $z_1 \in J(f)$, $U \ni z_1$. Then $f^n(U) \ni z_0$ for some n , so a preimage of z_0 lies in U .

Cor. $J(f)$ is a perfect set (no isolated pts)

Proof. $|J(f)| = \infty$ since either $J(f) \subset E(f) \subset F(f)$.

Then $J(f)$ has a limit pt z_0 , and the preimage of z_0 are a dense set of nonisolated pts.

Cor. $J(f)$ is separable

Prop. Either $J(f)$ is connected or has infinitely many connected components.

$$J_0 \stackrel{\text{finite}}{\subset} J / \stackrel{\text{infinite}}{\subset} J$$

Proof. Suppose $J(f) = J_0 \cup J_1$ is disconnected.

To each $z \in J(f)$, assign $\sigma(z) = (\sigma_i(z))_{i \geq 0}$ by

$$\sigma_i(z) = \begin{cases} 0 & \text{if } f^i(z) \in J_0 \\ 1 & \text{if } f^i(z) \in J_1 \end{cases} \quad \begin{matrix} 0' = 1 \\ 1' = 0 \end{matrix}$$

• Same connected comp. $\Rightarrow \sigma$ same

$$J_{b_0 b_1 \dots b_n} := \{z \in \hat{\mathbb{C}} \mid \sigma(z) = (b_0, \dots, b_n, \dots)\}$$

$$U_{b_0 \dots b_n} := J_{b_0 \dots b_n} \cup F(f)$$

Claim $\exists$ or any $(b_0, \dots, b_n) \exists z_1 \neq z_2 \in J_{b_0 \dots b_n}$ s.t.
 $\sigma(z_1) \neq \sigma(z_2)$.

$\perp$ Contrary: $\sigma(z) = (b_0, \dots) \nmid z'_i \in J_{b_0 \dots b_n}$.

$(b_0, \dots)$ contains at any 0 or 1, so for any sequence $\{f^{u_i}\}_{i \geq 0}$, there is a subsequence $\{f^{u_{i_k}}\}_{k \geq 0}$ s.t.

$f^{u_{i_k}}(U_{b_0 \dots b_n}) \subset U_0$ or U_1 . All of them can happen, so this is a real fun, and we could find a further subsequence that conv. loc. inf. But $J_{b_0 \dots b_n} \subset U_{b_0 \dots b_n}$.

$\Rightarrow$ Uncountably many distinct sequences.

Thm. (Baire Category) In a complete metric space, the countable $\cap$ of open dense sets is dense.

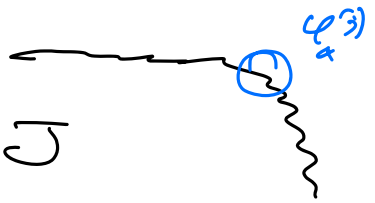
Prop. For a dense subset in $J(f)$,

$\{z, f(z), f^2(z), \dots\}$ is dense in $J(f)$.

Proof. $\{U_k^{(j)}\}_k$ has an open cover of $J(f)$ by open balls of radius $\frac{1}{j}$.

$V_k^{(j)} := \bigcup_{n \geq 0} f^{-n}(U_k^{(j)}) \cap J$ is dense open in J

Point in $\bigcap_{j, k} V_k^{(j)}$ have dense forward orbit. Baire.



Smooth Julia Set

Ex: $f(z) = z^n$, $J(f) = \{ |z| = 1 \}$

$\perp A \in \text{PSL}(2, \mathbb{C})$, $J(A \circ f \circ A^{-1}) = A \cdot J(f)$
 $F(A \circ f \circ A^{-1}) = A \cdot F(f)$

Ex: $f(z) = -z^2$, $A = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$, $(A \circ f \circ A^{-1})(z) = \frac{z^2 - 1}{2z}$
 $J(f) = \{ |z| = 1 \}$, $\mathbb{D} \rightarrow \mathbb{H}$, $J(g) = \mathbb{R}$

Newton's Method: To find the root of h , iterate

$N_h(z) = z - \frac{h(z)}{h'(z)}$. If $h(z) = z^2 + 1$, $N_h(z) = g(z)$.

Characterize Julia sets for real values.

$A(i)$

Ex: $f(z) = z^2 - 2$, $J(f) = [-2, 2]$

$A(-i)$ $\cdot -i$ $\xrightarrow{z^2-2}$

$z_0 \in [-2, 2] \Rightarrow f(z_0) \in [-2, 2]$, $[-2, 2]$ is forward invariant.

$z_0 \in [-2, 2] \Rightarrow \pm \sqrt{2 + z_0} \in [-2, 2]$, $[-2, 2]$ is backward invariant.

H.W. If $|x| \geq 3$, X is closed, and X is backward invariant, then $X \supset J(f)$.

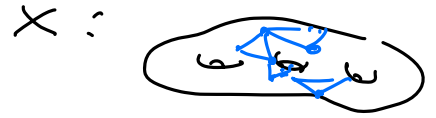
$\Rightarrow J(f) \subset [-2, 2] \Rightarrow F(f)$ is compact, $F(f) = \mathbb{H}(\infty)$

Since $[-2, 2] \not\rightarrow$ cs under f , $[-2, 2] \subset J(f)$.

Thm (Hawthorne 1995) All manifold Julia sets are, up to Möbius, either closed interval, circle, or union of

Haus. Dim > 1 (finite), or $\hat{\mathbb{C}}$. $\leftarrow$ For 2-dim. see
 general thm!

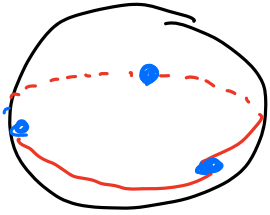
Def. Euler characteristic



$$\chi(X) = \#V - \#E + \#T, \quad \text{does not depend on triang.}$$



• $\hat{\mathbb{C}} = S^2$: $\chi = 2$ • $T^2 = \mathbb{C}/\Lambda$ $\chi = 0$

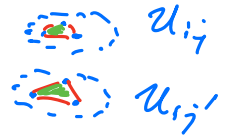


Prop. $\pi: X \rightarrow Y$ covering map of deg d
 $\Rightarrow \chi(X) = d \cdot \chi(Y)$

Proof. Pick open cover of Y by $\{V_i\}$ s.t.

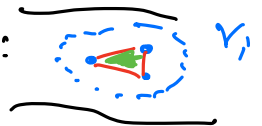
$$\pi: U_{ij} \xrightarrow{\sim} V_i \text{ for each comp } U_{ij} \text{ of } \pi^{-1}(V_i).$$

Now pick a triangulation for U_{ij} so that each vertex, line Δ lies inside a V_i .



Copy the triangulation under π^{-1} , $\pi \downarrow$

since it is a local home. Then Y :



one U times as many $\Rightarrow \chi(X) = d \cdot \chi(Y)$.

• $X \ni Y$ compact RS, $f: X \rightarrow Y$ lob. At each $x_0 \in X$,

$$f(x) = x^{k_{x_0}} h(x) \text{ for } h \neq 0 \text{ lob. } k_{x_0} \leftarrow \text{Racification degree}$$

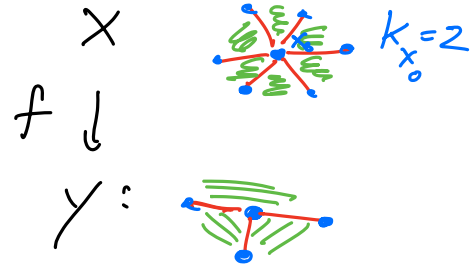
• if $f: X \rightarrow Y$ in degree d , then $\forall y \in Y$,

$$\sum_{f^{-1}(y)=x} k_x = d.$$

Thm. (Riemann-Hurwitz Formula) X, Y compact R.S., $f: X \rightarrow Y$ isb, degree d .

$$\chi(X) = d \cdot \chi(Y) - \sum_{x \in X} (k_x - 1)$$

Proof. Pick a small triangulation of Y so that at least one vertex at each $\{y \in Y \mid |f^{-1}(y)| < d\}$.



The principle of triangulation:

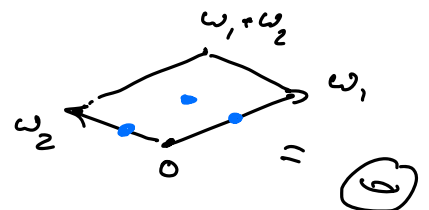
$$\begin{aligned} (\#V)(x) &= \sum_{\substack{\text{vertex} \\ y \in Y}} |f^{-1}(y)| = \sum_{\substack{\text{vertex} \\ y \in Y}} (d - \sum_{f(x)=y} (k_x - 1)) \\ &= d \cdot (\#V)(Y) - \sum_{x \in X} (k_x - 1) \end{aligned}$$

• Counting up for non. pts, we

$$(\#E)(x) = d \cdot (\#E)(Y), \quad (\#F)(x) = d \cdot (\#F)(Y).$$

Ex: $\Lambda = \mathbb{Z} \langle \omega_1, \omega_2 \rangle \subset \mathbb{C}$ lattice

$$g(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$



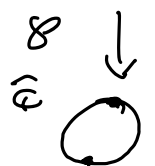
meromorphic function $\mathbb{C}$

$$g(z + \lambda) = g(z) \Rightarrow g: \mathbb{C}/\Lambda \rightarrow \hat{\mathbb{C}}$$

$$g(-z) = g(z), \quad g'(-z) = -g'(z)$$

$\Rightarrow$ double pole at 0, serial part of deg 2 at $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1 + \omega_2}{2}$

$$k_0 = k_{\omega_1/2} = k_{\omega_2/2} = k_{\frac{\omega_1 + \omega_2}{2}} = 2,$$



$$R-H: \chi(\mathbb{C}/\Lambda) = d \cdot \chi(\hat{\mathbb{C}}) - \sum (k_x - 1)$$

$$0 = 2d - 4 \rightarrow d = 2$$

$$\Rightarrow f(z) = f(z') \text{ iff } z = -z' + \lambda$$

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{f_\Lambda} & \mathbb{C}/\Lambda \\ \downarrow \delta & & \downarrow \delta \\ \hat{\mathbb{C}} & \xrightarrow{f_\Lambda} & \hat{\mathbb{C}} \end{array} \quad \begin{array}{l} f(z) = f(z') \Rightarrow z = -z' + \lambda \\ 2z = -2z' + 2\lambda \\ f(2z) = f(2z') \end{array}$$

$$f_\Lambda(f(z)) = f(2z), \quad f_\Lambda \text{ is deg } 4 \quad (\text{Zurich map})$$

Prop. $J(f_\Lambda) = \hat{\mathbb{C}}$

$$P-f. 1) \left\{ \frac{p_1}{q_1} \omega_1 + \frac{p_2}{q_2} \omega_2 \in \mathbb{C}/\Lambda \mid q_1, q_2 \text{ odd} \right\} \text{ is dense in } \mathbb{C}/\Lambda$$

and are all preimages for $f: \mathbb{C}/\Lambda \rightarrow \mathbb{C}/\Lambda$.

$$2) f_\Lambda^n(f(z)) = f(f^n(z))$$

$$\cdot f(z_0) = z_0 \Rightarrow f_\Lambda^n(f(z_0)) = f(z_0)$$

$$\cdot (f_\Lambda^n)'(f(z_0)) f'(z_0) = f'(z_0) (f^n)'(z_0) = f'(z_0) 2^n$$

$$\Rightarrow \lambda = 2^n \text{ multiplicity}$$

$\hat{\mathbb{C}}$ has a dense set of rep. per. pts

* $J(f) = \hat{\mathbb{C}}$ for other f

Postcont. finite
Project ideas